Colors and Invariants

Pleasanton Math Circle: Middle School

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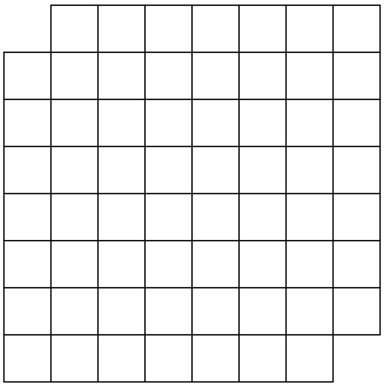
§1 Warm-Up

Problem 1.1. Suppose that I completely color one side of a sheet of paper in two colors, so that any point on this page will be one of the two colors. Prove that no matter how I color this piece of paper, I can always find two points an inch apart that are the same color. (Assume that the length and width of the page are much larger than 1 inch)

Problem 1.2. We take an 8×8 chessboard and remove two opposite corners like so:



We have 31 dominoes, and each domino is a 1×2 rectangle that can completely cover two neighboring squares of the chessboard. Is it possible to completely cover this chessboard with these 31 dominoes?

§2 Colors

The following problems involve colors. If the problem does not specifically mention the use of colors, then try to find a way to assign colors to different parts of the problem. For example, if the problem is about a chessboard, then try to assign colors to the squares. (In the more difficult problems, you may have to be more clever in how you assign colors!)

Problem 2.1. In every small cell of a 5×5 chess board sits a bug. At certain moment all the bugs crawl to neighboring (via a horizontal or a vertical edge) cells. Will there always be some cell that becomes empty?

Problem 2.2. Prove that it is impossible to cut a 10×10 chess board into 1×4 rectangles.

Problem 2.3. A rectangle is tiled with tiles of size 2×2 and 1×4. The tiles had been removed from the rectangle, and in the process one tile of size 2×2 was lost. We replaced it with a tile of size 1×4. Prove that it is impossible to tile the rectangle with the resulting collection of tiles.

Problem 2.4. Consider Problem 1.2. This time, prove that I can always find two points an inch apart that are the same color if you have three colors to choose from.

§3 Invariants

Definition 3.1 (Invariant)**.** An invariant is a property that doesn't change after performing an operation. Try to spot an invariant in the following examples:

- We are given a polygon on the plane. An operation is defined as either rotating, translating, or reflecting it. What stays the same? (There are many answers)
- A board has a bunch of numbers on it. An operation is defined as taking two numbers on the board and replacing them with the square root of their product. For example, 6 and 7 would be replaced with $\sqrt{42}$ and $\sqrt{42}$. What stays the same?

Problem 3.2. You will fill in each square in the following expression with either a $+$ or $-$ sign:

$$1\square 2\square 3\square 4\square 5\square 6\square 7\square 8\square 9\square 10$$

Prove that no matter how you choose to fill in the squares, this expression cannot equal 0.

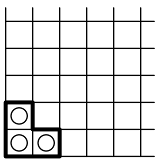
Problem 3.3. We write the numbers from 1 through 10 on a board. An operation consists of taking two numbers, say a and b, erasing them, and replacing them with $ab + a + b$. If we repeat this until there is one number left, what is this number?

Problem 3.4. Seven vertices of a cube are labeled 0, and the remaining vertex labeled 1. You're allowed to change the labels by picking an edge of the cube, and adding 1 to the labels of both of its endpoints. After repeating this multiple times, can you make all labels divisible by 3?

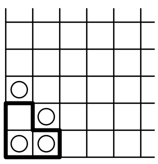
Problem 3.5. Challenge: Show that when a 6×6 square floor is tiled using 1×2 rectangular tiles, there is always a straight line which crosses the floor without cutting through any of the tiles.

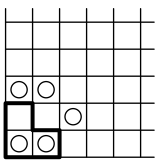
Problem 3.6. Challenge: A 10×10 square field is divided into 100 equal square patches, 9 of which are overgrown with weeds. It is known that during a year the weeds spread to those patches that have no less than two neighboring (i.e., having a common side) patches that are already overgrown with weeds. If a patch does not have at least two neighboring patches with weeds, then weeds will not spread to that patch. Prove that the field will never overgrow completely with weeds.

Problem 3.7. "Three Prisoners." Challenge:



Consider a grid that extends infinitely in the rightward and upward directions. At the start, three dots or "prisoners" are in the three cells in the bottom-left corner. These prisoners want to escape the prison (i.e. the outlined region), and they can move in the following way: if both the cell above and the cell to the right are empty, the prisoner can divide into two and move to those cells. Basically, replace the existing prisoner and fill in the cells above it and to the right of it. The two diagrams below show a possible sequence of two moves you can start with:





Is it possible for all prisoners to escape the prison? That is, is it possible for the three cells at the bottom-left to be empty? Once you have an answer, can you prove it?